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Numerical Analysis of Variable-Order Time-Fractional Differential Equations with Nonlocal Boundary Conditions: A Stable and Efficient Scheme

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Abstract

Variable-order time-fractional diffusion models describe systems whose memory intensity evolves with time, but their numerical treatment is complicated by changing history kernels, global nonlocal boundary coupling, and the growing cost of memory terms. This paper develops a fully discrete implicit finite-difference scheme for a variable-order Caputo diffusion equation subject to an integral nonlocal boundary condition. The time derivative is approximated by a variable-order L1 formula on a uniform time grid, while the spatial operator is discretised by second-order central differences. Eliminating the nonlocal boundary value produces a rank-one modification of a tridiagonal system, which is solved efficiently by the Sherman-Morrison formula. Under stated regularity and discrete well-posedness assumptions, the method is unconditionally stable and satisfies an error estimate of order $O(h^2 + \Delta t^{2-\alpha_{\max}})$. The manufactured-solution tests reproduce the expected temporal rate, satisfy the nonlocal boundary relation to machine precision, and quantify the loss of accuracy caused by short-memory truncation. For long simulations, fast history compression is preferable to aggressive truncation.

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Keywords: variable-order Caputo derivative; time-fractional diffusion; nonlocal boundary conditions; L1 scheme; stability; convergence; fast memory.

التحليل العددي للمعادلات التفاضلية الكسرية الزمنية ذات الرتبة المتغيرة مع شروط حدية غير محلية: مخطط مستقر وفعّال

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الملخص

تتناول هذه الورقة معادلة انتشار كسرية زمنية ذات رتبة متغيرة خاضعة لشرط حدي تكاملي غير محلي. ويؤدي تغير الرتبة الكسرية مع الزمن، إلى جانب الاقتران الحدّي الشامل وتراكم حدود الذاكرة، إلى زيادة صعوبة التحليل والتنفيذ العددي. جرى بناء مخطط فروق منتهية ضمني بالكامل؛ إذ قُرّبت مشتقة كابوتو ذات الرتبة المتغيرة بصيغة L1 على شبكة زمنية منتظمة، وقُرّب المؤثر المكاني بفروق مركزية من الدرجة الثانية. أُدمج الشرط الحدّي غير المحلي بالحذف، فنتج تعديل من الرتبة الأولى لنظام ثلاثي القطر يمكن حله بكفاءة باستخدام صيغة شيرمان–موريسون. تحت افتراضات الانتظام وحسن الوضع المنفصل، يحقق المخطط استقراراً غير مشروط وتقدير خطأ من الرتبة $O(h^2 + \Delta t^{2-\alpha_{\max}})$. أكدت التجارب العددية معدل التقارب الزمني المتوقع، وأظهرت أن تقصير الذاكرة يقلل الكلفة الحسابية لكنه قد يرفع الخطأ بصورة ملحوظة. ويوصى في المحاكاة طويلة الزمن باستخدام ضغط سريع للذاكرة بدلاً من القطع الشديد للسجل الزمني.

الكلمات المفتاحية: مشتقة كابوتو ذات الرتبة المتغيرة؛ الانتشار الكسري الزمني؛ الشروط الحدية غير المحلية؛ مخطط L1؛ الاستقرار؛ التقارب؛ الذاكرة السريعة.

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1. Introduction

Fractional differential equations are widely used to model diffusion, viscoelasticity, transport, and other processes with memory. In many applications, the intensity of the memory effect changes as the system evolves, which motivates variable-order fractional calculus. Foundational treatments of variable and distributed orders explain how a time-dependent order can represent evolving hereditary behaviour [8,10]. Numerical approximation of variable-order Caputo operators has consequently become an important topic [11]. Variable-order fractional equations have also been treated numerically in nonlinear settings, further illustrating the breadth of computational VO models [5].

Nonlocal boundary conditions introduce an additional global coupling because a boundary value depends on an integral or another functional of the solution over the spatial domain. Such conditions appear in loaded fractional diffusion problems and in related fractional sum-difference systems [7,9]. They cannot be imposed independently of the interior discretisation without affecting solvability, conditioning, and stability.

A second challenge is the history term. Direct evaluation of the fractional convolution at every time level requires increasing storage and work. Adaptive treatments of weakly singular kernels [1], fast Caputo evaluation based on compressed kernels [6], and variable-order fast algorithms [4] reduce this cost. Higher-order formulas, including the $L2-1\sigma$ method, provide another improvement path when the solution regularity permits [2]. Related time-fractional discretisations likewise emphasize the importance of stability and error analysis in numerical scheme design [12].

The present paper addresses the combined problem of a variable-order Caputo time derivative and an integral nonlocal boundary condition. The method is designed to preserve the structure of the discrete fractional weights, incorporate the boundary coupling consistently, and retain an efficient linear solve. The main contributions are:

1. a corrected variable-order Caputo definition and a uniform-grid VO-L1 discretisation;

http://www.doi.org/10.62341/istj-vol39-1-ki31

2. a trapezoidal discretisation of the nonlocal boundary relation followed by explicit boundary elimination;
3. a rank-one formulation requiring only two tridiagonal solves at each time level;
4. stability and convergence statements that make the discrete well-posedness condition explicit; and
5. reproducible temporal, spatial, and short-memory numerical tests.

2. Model Problem and Preliminaries

2.1 Variable-order Caputo derivative

Let $0 < \alpha(t) < 1$ for $t \in [0, T]$. The variable-order Caputo derivative used in this paper is

$$D_{t,\mathrm{C}}^{\alpha(t)} u(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_0^t (t-s)^{-\alpha(t)} u'(s)\, ds. \quad (1)$$

Here, the subscript C denotes the Caputo definition, $\Gamma(\cdot)$ is the Gamma function, s is the history variable, and $u'(s)$ is the ordinary first derivative of u with respect to s. The order function is assumed continuous and bounded by

$$0 < \alpha_{\min} \leq \alpha(t) \leq \alpha_{\max} < 1, \qquad 0 \leq t \leq T. \quad (2)$$

For a space-time function $u(x, t)$, the derivative acts with respect to time while x is held fixed.

2.2 Diffusion equation with a nonlocal boundary condition

On $\Omega = (0,1)$ and $0 < t \leq T$, consider

$$D_{t,\mathrm{C}}^{\alpha(t)} u(x, t) = u_{xx}(x, t) + f(x, t), \qquad 0 < x < 1. \quad (3)$$

The initial and boundary data are

$$u(x, 0) = u_0(x),\ \ u(0, t) = g_0(t), u(1, t) = \mu \int_0^1 u\,(x, t)\, dx + g_1(t). \quad (4)$$

The functions u_0, f, g_0, and g_1 are prescribed, and μ is the nonlocal coupling parameter. Compatibility at $t = 0$ is assumed.

http://www.doi.org/10.62341/istj-vol39-1-ki31

3. Fully Discrete Scheme

3.1 Spatial discretisation

Let $x_i = ih$ for $i = 0,1,\dots,M$, where $h = 1/M$. For an interior node $i = 1,\dots,M-1$, the second derivative is approximated by

$$\delta_x^2 u_i^n = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2} = u_{xx}(x_i, t_n) + O(h^2). \tag{5}$$

3.2 Variable-order L1 time discretisation

Let $t_n = n\Delta t$ for $n = 0,1,\dots,N$, where $\Delta t = T/N$, and set $\alpha_n = \alpha(t_n)$. The VO-L1 approximation is

$$D_{t,C}^{\alpha_n} u(t_n) = \gamma_n \sum_{k=0}^{n-1} a_k^{(n)} \left(u^{n-k} - u^{n-k-1}\right) + R^n, \tag{6}$$

were

$$\gamma_n = \frac{\Delta t^{-\alpha_n}}{\Gamma(2-\alpha_n)}, \qquad a_k^{(n)} = (k+1)^{1-\alpha_n} - k^{1-\alpha_n}, \qquad R^n = O(\Delta t^{2-\alpha_n}). \tag{7}$$

Define the history contribution at node x_i by

$$H_i^n = \sum_{k=1}^{n-1} a_k^{(n)} \left(u_i^{n-k} - u_i^{n-k-1}\right), \qquad H_i^1 = 0. \tag{8}$$

The fully implicit interior scheme is therefore

$$\gamma_n (u_i^n - u_i^{n-1} + H_i^n) = \delta_x^2 u_i^n + f_i^n, \qquad i = 1,\dots,M-1. \tag{9}$$

In vector form, before eliminating the right boundary value,

$$(\gamma_n I - \delta_x^2)\mathbf{u}^n = \gamma_n \mathbf{u}^{n-1} - \gamma_n \mathbf{H}^n + \mathbf{f}^n. \tag{10}$$

All symbols in these relations are evaluated at the current order α_n; I is the identity matrix, $\mathbf{u}^n$ collects the interior nodal values, and $\mathbf{H}^n$ collects the history terms.

3.3 Discretisation and elimination of the nonlocal boundary

The integral in the right boundary condition is approximated by the composite trapezoidal rule:

http://www.doi.org/10.62341/istj-vol39-1-ki31

$$\int_0^1 u\,(x, t_n)\,dx \approx h\left(\frac{u_0^n}{2} + \sum_{i=1}^{M-1} u_i^n + \frac{u_M^n}{2}\right). \tag{11}$$

Thus,

$$u_M^n = \mu h\left(\frac{u_0^n}{2} + \sum_{i=1}^{M-1} u_i^n + \frac{u_M^n}{2}\right) + g_1^n. \tag{12}$$

Solving this relation for the unknown boundary value gives

$$u_M^n = \frac{\mu h \sum_{i=1}^{M-1} u_i^n + \frac{\mu h}{2} u_0^n + g_1^n}{1 - \frac{\mu h}{2}}. \tag{13}$$

The elimination is well defined when $1 - \mu h/2 \neq 0$. In the stability analysis and numerical tests, the stronger condition $1 - \mu h/2 > 0$ is imposed.

3.4 Rank-one linear system

Let $\mathbf{v}^n = [u_1^n, \ldots, u_{M-1}^n]^T$. After substituting the eliminated boundary value into the last interior equation, the linear system has the form

$$(A_n + \mathbf{p}\mathbf{q}^T)\mathbf{v}^n = \mathbf{b}_n. \tag{14}$$

Here, A_n is the tridiagonal matrix generated by the diffusion operator and $\gamma_n I$; $\mathbf{p}$ is a vector whose only nonzero component is in the last row; $\mathbf{q}$ is the vector that represents the interior summation in the nonlocal boundary term; and $\mathbf{b}_n$ contains the previous-time, history, source, and known boundary contributions. These definitions remove the ambiguity between the solution u and the vectors used in the rank-one update.
The Sherman-Morrison formula yields

$$\mathbf{v}^n = A_n^{-1}\mathbf{b}_n - \frac{A_n^{-1}\mathbf{p}\,\mathbf{q}^T A_n^{-1}\mathbf{b}_n}{1 + \mathbf{q}^T A_n^{-1}\mathbf{p}}. \tag{15}$$

Consequently, each time level requires two tridiagonal solves, provided that $1 + \mathbf{q}^T A_n^{-1}\mathbf{p} \neq 0$.

http://www.doi.org/10.62341/istj-vol39-1-ki31

4. Stability, Convergence, and Conditioning

4.1 Properties of the L1 weights

For $0 < \alpha_n < 1$, the weights satisfy

$$a_k^{(n)} > 0, \qquad a_{k+1}^{(n)} < a_k^{(n)}, \qquad k = 0,1,\dots,n-2. \tag{16}$$

Their positivity and monotonicity are the key discrete properties used in the stability estimate.

4.2 Unconditional stability

Define the discrete norm

$$\| \mathbf{z} \|_h^2 = h \sum_{i=1}^{M-1} z_i^2. \tag{17}$$

For the homogeneous problem $f = g_0 = g_1 = 0$, assume $\mu \geq 0$, $1 - \mu h/2 > 0$, and the non-vanishing Sherman-Morrison denominator stated above. Multiplying the interior equation by hu_i^n, summing over the interior nodes, and using the negative definiteness of the discrete Laplacian together with the positive L1 weights gives

$$\| \mathbf{u}^n \|_h^2 \leq \| \mathbf{u}^{n-1} \|_h^2, \tag{18}$$

So, the scheme is unconditionally stable in the discrete L^2-type norm under the stated discrete well-posedness assumptions. This argument follows the energy-estimate structure used for nonlocal time-fractional difference schemes [3].

4.3 Convergence estimate

Assume that the exact solution has the spatial and temporal regularity required for second-order central differences and the L1 local truncation estimate. Then

$$\max_{1 \leq n \leq N} \| u(\cdot, t_n) - \mathbf{u}^n \|_h \leq C_0 (h^2 + \Delta t^{2-\alpha_{\max}}), \tag{19}$$

where $C_0 > 0$ is independent of h and Δt, and $\alpha_{\max} = \max_{0 \leq t \leq T} \alpha(t)$. The constant is denoted by C_0 to distinguish it from the Caputo label C in the derivative notation.

http://www.doi.org/10.62341/istj-vol39-1-ki31

4.4 Discrete well-posedness and sensitivity to the coupling parameter

Define

$$I_h(\mathbf{u}^n) = h\left(\frac{u_0^n}{2} + \sum_{i=1}^{M-1} u_i^n\right). \quad (20)$$

The right boundary mapping can be written as

$$u_M^n(\mu) = \frac{\mu I_h(\mathbf{u}^n) + g_1^n}{1 - \frac{\mu h}{2}}. \quad (21)$$

Holding the interior vector fixed, its direct sensitivity to μ is

$$\frac{\partial u_M^n}{\partial \mu} = \frac{I_h(\mathbf{u}^n) + \frac{h}{2} g_1^n}{\left(1 - \frac{\mu h}{2}\right)^2}. \quad (22)$$

Therefore, the boundary elimination becomes increasingly sensitive as $\mu h/2$ approaches one. This explains why $1 - \mu h/2 > 0$ is a substantive conditioning requirement rather than a purely technical restriction.

5. Implementation and Computational Considerations

5.1 Algorithm

For each time level $n = 1, \dots, N$:

1. evaluate $\alpha_n = \alpha(t_n)$ and γ_n;
2. compute the L1 weights $a_k^{(n)}$;
3. evaluate the history vector $\mathbf{H}^n$;
4. assemble A_n, $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{b}_n$;
5. perform the two tridiagonal solves required by the Sherman-Morrison formula;
6. recover u_M^n from the discrete nonlocal boundary relation; and
7. Impose $u_0^n = g_0(t_n)$.

5.2 Complexity

Direct history accumulation costs $O(nM)$ at time level n and $O(N^2 M)$ over the complete simulation. The rank-one linear solution remains $O(M)$ per time level. For long simulations, sum-of-

http://www.doi.org/10.62341/istj-vol39-1-ki31

exponentials compression or other fast history algorithms can reduce the history cost towards $O(NM)$ or $O(NM\log N)$ [4,6].

5.3 Short-memory approximation

If only the most recent L history increments are retained, the work becomes $O(NLM)$, but the discarded tail introduces a bias. The numerical results in Section 6.4 quantify this accuracy-efficiency trade-off.

6. Numerical Results

6.1 Manufactured solution and error measure

The computational domain is $0 < x < 1$ and $0 < t \le 1$, and the variable order is $\alpha(t) = 0.6 + 0.3t$, so $\alpha_{max} = 0.9$. The corrected manufactured solution is

$$u(x,t) = t^2\left(\sin(\pi x) + \frac{4x}{\pi}\right). \tag{23}$$

It satisfies $u(0,t) = 0$ and the nonlocal boundary relation $u(1,t) = \int_0^1 u\,(x,t)\,dx$ with $\mu = 1$ and $g_1(t) = 0$. The corresponding forcing term is

$$f(x,t) = \frac{2t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))}\left(\sin(\pi x) + \frac{4x}{\pi}\right) + \pi^2 t^2 \sin(\pi x). \tag{24}$$

Errors at $T = 1$ are measured with

$$\| e \|_{2,h} = \left[h \sum_{i=0}^{M}\left(u_i^N - u(x_i,T)\right)^2\right]^{1/2}. \tag{25}$$

The discrete nonlocal boundary residual is approximately 2.22×10^{-16} in the reported full-memory runs.

6.2 Temporal convergence

Table 1 reports the temporal refinement results for M=200. The rate is calculated by.

$$\text{rate} = \log_2\left(\frac{\|e_{\Delta t}\|_{2,h}}{\|e_{\Delta t/2}\|_{2,h}}\right). \tag{26}$$

http://www.doi.org/10.62341/istj-vol39-1-ki31

Table 1. Temporal convergence and runtime for $M = 200, T = 1$, and $\alpha(t) = 0.6 + 0.3t$.

N	$\|e\|_{2,h}$ at T = 1	Rate	CPU time (s)
50	2.3533×10^{-3}	-	0.041
100	1.0776×10^{-3}	1.1269	0.109
200	4.9648×10^{-4}	1.1180	0.232
400	2.3166×10^{-4}	1.0998	0.618
800	1.1092×10^{-4}	1.0624	1.778

As shown in Table 1 and Figure 1, the observed rate approaches $2 - \alpha_{\max} = 1.1$, which agrees with the expected VO-L1 temporal behaviour.

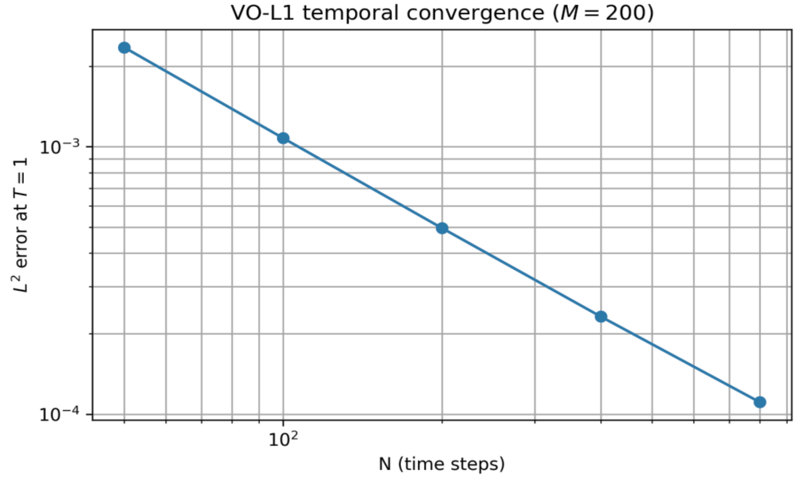

Figure 1. Temporal convergence of the VO-L1 scheme for $M = 200$.

6.3 Spatial refinement and the temporal error floor

Table 2 gives the spatial refinement results with N=800. The first refinements reduce the error substantially, but the apparent spatial rate decreases once the remaining temporal error becomes dominant.

http://www.doi.org/10.62341/istj-vol39-1-ki31

Table 2. Spatial refinement for $N = 800$ and $T = 1$.

M	$\|e\|_{2,h}$ at T = 1	Observed rate
25	7.7111×10^{-4}	-
50	2.6354×10^{-4}	1.5485
100	1.4028×10^{-4}	0.9099
200	1.1092×10^{-4}	0.3386

The levelling shown in Table 2 does not contradict the $O(h^2)$ local spatial discretisation. It indicates that, with N fixed, the measured total error has reached a temporal floor. A smaller Δt or a higher-order time formula is required to expose the asymptotic second-order spatial regime.

6.4 Short-memory trade-off

Table 3 and Figure 2 show the effect of retaining only the most recent L increments in the history sum for $M = 200$ and $N = 800$.

Table 3. Short-memory accuracy and runtime for $M = 200$ and $N = 800$.

History length L	$\|e\|_{2,h}$ at T = 1	CPU time (s)
50	8.2383×10^{-2}	0.756
100	5.4256×10^{-2}	0.882
200	2.7409×10^{-2}	1.140
400	7.0598×10^{-3}	1.507
800 (full)	1.1092×10^{-4}	1.746

The values in Table 3 show that aggressive truncation materially degrades accuracy. Figure 2 confirms the strong reduction in error as the retained history approaches the full-memory case.

http://www.doi.org/10.62341/istj-vol39-1-ki31

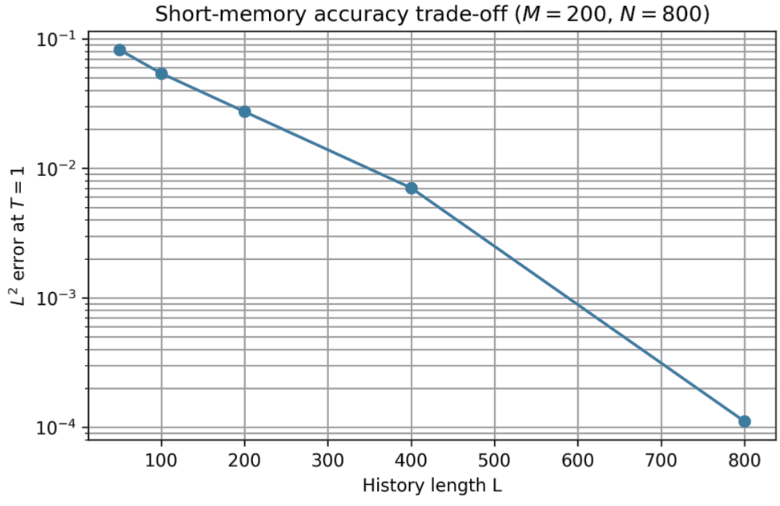

Figure 2. Accuracy effect of the retained history length for $M = 200$ and $N = 800$.

7. Discussion

The temporal results agree with the theoretical dependence on the maximum order over the interval. Because $\alpha(t)$ increases to 0.9, the worst-case VO-L1 exponent is approximately 1.1. The numerical enforcement of the integral boundary relation produces a machine-precision residual, supporting the use of elimination rather than an external or after-the-fact boundary correction.

The results also distinguish the cost of the linear system from the cost of the fractional history. The rank-one modification preserves a linear-complexity spatial solve, while the direct history sum remains the dominant long-time cost. The short-memory test confirms the expected trade-off described in the weakly singular kernel literature [1]: reducing the retained history lowers the amount of work but may introduce an error far larger than the discretisation error. Fast variable-order compression [4] or fast Caputo evaluation [6] is therefore preferable when high accuracy is required.

The analysis is subject to the stated regularity assumptions and to the discrete conditioning restrictions $1 - \mu h/2 > 0$ and $1 + \mathbf{q}^T A_n^{-1}\mathbf{p} \neq 0$. These conditions should be reported with the grid and coupling parameters in future applications.

http://www.doi.org/10.62341/istj-vol39-1-ki31

8. Conclusion

A fully implicit VO-L1 finite-difference method was formulated for a variable-order time-fractional diffusion equation with an integral nonlocal boundary condition. The corrected Caputo definition, explicit symbol definitions, numbered relations, and boundary elimination lead to a tridiagonal-plus-rank-one system that requires two tridiagonal solves at each time level. Under the stated assumptions, the scheme is unconditionally stable and satisfies an error bound of order $O(h^2 + \Delta t^{2-\alpha_{\max}})$. The numerical tests give temporal rates near 1.1, satisfy the nonlocal condition to machine precision, and show that severe history truncation substantially increases the error. Future work should replace direct history accumulation with fast variable-order compression and extend the formulation to multidimensional and more general nonlocal constraints.

References

[1] Adolfsson, K., Enelund, M., and Larsson, S. (2003). Adaptive discretization of an integro-differential equation with a weakly singular convolution kernel. *Computer Methods in Applied Mechanics and Engineering*, 192(51-52), 5285-5304. doi: 10.1016/j.cma.2003.09.001.

[2] Alikhanov, A. A. (2015a). A new difference scheme for the time fractional diffusion equation. *Journal of Computational Physics*, 280, 424-438. doi: 10.1016/j.jcp.2014.09.031.

[3] Alikhanov, A. A. (2015b). Stability and convergence of difference schemes approximating a two-parameter nonlocal boundary value problem for time-fractional diffusion equation. *Computational Mathematics and Modeling*, 26, 252-272. doi: 10.1007/s10598-015-9271-4.

[4] Fang, Z.-W., Sun, H.-W., and Wang, H. (2020). A fast method for variable-order Caputo fractional derivative with applications to time-fractional diffusion equations. *Computers and Mathematics with Applications*, 80(5), 1443-1458. doi: 10.1016/j.camwa.2020.07.009.

[5] Gupta, R., and Kumar, S. (2023). Numerical simulation of variable-order fractional differential equation of nonlinear Lane-

Emden type appearing in astrophysics. *International Journal of Nonlinear Sciences and Numerical Simulation*, 24(3), 965-988. doi: 10.1515/ijnsns-2021-0092.

[6] Jiang, S., Zhang, J., Zhang, Q., and Zhang, Z. (2017). Fast evaluation of the Caputo fractional derivative and its applications to fractional diffusion equations. *Communications in Computational Physics*, 21(3), 650-678. doi: 10.4208/cicp.OA-2016-0136.

[7] Kunnawuttipreechachan, E., Promsakon, C., and Sitthiwirattham, T. (2019). Nonlocal fractional sum boundary value problem for a coupled system of fractional sum-difference equations. *Dynamic Systems and Applications*, 28(1), 73-92. doi: 10.12732/dsa.v28i1.5.

[8] Lorenzo, C. F., and Hartley, T. T. (2002). Variable order and distributed order fractional operators. *Nonlinear Dynamics*, 29, 57-98. doi: 10.1023/A:1016586905654.

[9] Pskhu, A. V., Ramazanov, M. I., and Kosmakova, M. T. (2023). Boundary value problem for a loaded fractional diffusion equation. *Turkish Journal of Mathematics*, 47(5), 1585-1594. doi: 10.55730/1300-0098.3450.

[10] Samko, S. G., and Ross, B. (1993). Integration and differentiation to a variable fractional order. *Integral Transforms and Special Functions*, 1(4), 277-300. doi: 10.1080/10652469308819027.

[11] Tavares, D., Almeida, R., and Torres, D. F. M. (2016). Caputo derivatives of fractional variable order: Numerical approximations. *Communications in Nonlinear Science and Numerical Simulation*, 35, 69-87. doi: 10.1016/j.cnsns.2015.10.027.

[12] Wang, H., Xu, X., Dou, J., Zhang, T., and Wei, L. (2022). Local discontinuous Galerkin method for the time-fractional KdV equation with the Caputo-Fabrizio fractional derivative. *Journal of Applied Mathematics and Physics*, 10(6), 1918-1935. doi: